NUST Journal of Engineering Sciences, Vol. 14, No. 1, 2021

Natural Frequencies of Exponential Functionally Graded Beams based on Classical Beam Theory

Lan Hoang That Ton [*], Nam Phuong Thi Ngo
Department of Civil Engineering, University of Architecture HCMC, 196 Pasteur Street, District 3, HCM City, Vietnam
[*]Corresponding Author: tltechonlinesom@gmail.com

Abstract

In this paper, the natural frequencies of exponential functionally graded beams are determined from two separate formulations, one based on an analytical approach and the other based on a numerical approach. The classical beam theory is carried out with various boundary conditions. The results obtained in this paper are presented and compared with other results in the references to verify the correctness in implementing the formula and writing the Matlab code. This paper can help researchers have an overview of the vibration characteristics of the exponential functionally graded beams. Furthermore, they can enhance their research by modifying more advanced materials such as beams reinforced by graphene platelets, beams' shape, etc. Last but not least, with the strong application of the functionally graded material in real life, it would be good if there were more data related to this issue.

Keywords: Natural frequency, exponential functionally graded material, beam, classical beam theory

Introduction

In the last few decades, functionally graded material has become one of the smart materials widely used in industry. The concept is to make a composite material by varying the microstructure from one material to another with a specific gradient. This enables the material to have the best behavior of both materials. If it is for thermal or corrosive resistance or malleability and toughness, both material strengths may be used to avoid troubles related to the above issues [1-5]. Due to the wide application of functionally graded material, various studies have been conducted on this material's thermal and mechanical behavior as [6-21]. Among three kinds of structures like beam, plate and shell, the beam has always been the interest of researchers because of its applications. The displacement field based on higher-order shear deformation theory was implemented to study the static behavior of functionally graded metal-ceramic (FGM) beams under ambient temperature by Kadoli *et al.* [5]. Using the principle of stationary potential energy, the finite element form of static equilibrium equation for the FGM beam was presented in this study.

Moreover, the higher-order theory was also extended to functionally graded beams with continuously varying material properties. With shear deformation taken into account, a single governing equation for an auxiliary function was derived from the basic equations of elasticity as in [6] by Li *et al.*. Based on the analytical way, some references are given to solve functionally graded beams by authors Zhong [7], Sankar [9], Khalili [10], etc. Especially, an efficient finite element model for vibration analysis of a non local Euler–Bernoulli beam has been reported by Eltaher *et al.* [13]. Furthermore, an analytical solution was developed to study the free vibration of exponential functionally graded beams with a single delamination by Li and Shu [14]. Euler–Bernoulli hypothesis, the '*free mode*' and '*constrained mode*' assumptions in delamination vibration were adopted in this article and so on.

The main objective of this work is to calculate the natural frequencies of an exponential functionally graded beam under four types of boundary conditions (BCs). Furthermore, this paper presents two different ways to get these values related to analytical solution and finite element strategy based on the classical beam theory. Although the topic and approach of this paper are not new, the authors' main aim is to reaffirm the applicability of classical beam theory once again to analyze the functionally graded beams with acceptable results.

This paper has four sections. Sect. 1 gives the introduction as above. Sect. 2 presents the formulations as well as Sect. 3 shows some actual results. Finally, a few comments are also given in Sect. 4, respectively.

Formulations

Exponential functionally graded material

The exponential functionally graded material is one of three kinds of functionally graded material as follows Figure 1. It is assumed that the material properties continuously change across the thickness of the beam according to the exponential function.

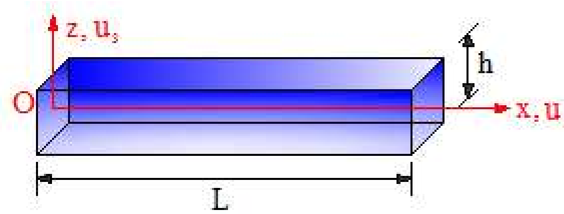

Fig. 1 The exponential functionally graded material.

$$E(z)=E_1 e^{\frac{1}{h}\ln\frac{E_1}{E_2}\left(z+\frac{h}{2}\right)} \tag{1}$$

$$\nu(z)=\nu_1 e^{\frac{1}{h}\ln\frac{\nu_1}{\nu_2}\left(z+\frac{h}{2}\right)} \tag{2}$$

$$\rho(z)=\rho_1 e^{\frac{1}{h}\ln\frac{\rho_1}{\rho_2}\left(z+\frac{h}{2}\right)} \tag{3}$$

where (E_1, E_2), (ν_1, ν_2) and (ρ_1, ρ_2) are the material properties like modulus of Young, Poisson ratio, density corresponding to the upper and lower surfaces of the exponential functionally graded beam.

Analytical approach

Based on the classical beam theory, the displacement field can be written:

$$u_1(x,z,t)=u_1^o(x,t)-z\frac{\partial u_3(x,t)}{\partial x} \tag{4}$$

$$u_3(x,z,t)=u_3^o(x,t) \tag{5}$$

The strains are given:

https://doi.org/10.24949/njes.v14i1.647

$$\varepsilon_{11}=\frac{\partial u_1(x,z,t)}{\partial x}=\frac{\partial u_1^o(x,t)}{\partial x}-z\frac{\partial^2 u_3^o(x,t)}{\partial x^2} \quad (6)$$

$$\varepsilon_{33}=\frac{\partial u_3(x,z,t)}{\partial z}=\frac{\partial u_3^o(x,t)}{\partial z} \quad (7)$$

$$\varepsilon_{13}=\frac{\partial u_1(x,z,t)}{\partial z}+\frac{\partial u_3(x,z,t)}{\partial x}= =-\frac{\partial u_3^o(x,t)}{\partial x}+\frac{\partial u_3^o(x,t)}{\partial x}=0 \quad (8)$$

The stresses can be expressed as:

$$\sigma_{11}=\frac{E(z)}{1-\nu^2(z)}\varepsilon_{11} \quad (9)$$

$$\sigma_{33}=\frac{E(z)}{1-\nu^2(z)}\varepsilon_{33} \quad (10)$$

$$\sigma_{13}=\frac{E(z)}{2(1+\nu(z))}\varepsilon_{13}=0 \quad (11)$$

Using the principle of least action, the equations of motion are obtained:

$$A_{11}\frac{\partial^2 u_1^o(x,t)}{\partial x^2}=B_{11}\frac{\partial^3 u_3^o(x,t)}{\partial x^3} \quad (12)$$

$$\frac{\partial^2 u_3^o(x,t)}{\partial t^2}=-\frac{1}{I_1}\left(D_{11}-\frac{B_{11}^2}{A_{11}}\right)\frac{\partial^4 u_3^o(x,t)}{\partial x^4} =\zeta\frac{\partial^4 u_3^o(x,t)}{\partial x^4} \quad (13)$$

in which:

$$A_{11}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{E(z)}{1-\nu^2(z)}dz \quad (14)$$

$$B_{11}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{E(z)}{1-\nu^2(z)}zdz \quad (15)$$

$$D_{11}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{E(z)}{1-\nu^2(z)}z^2dz \quad (16)$$

and

$$I_1=\int_{-\frac{h}{2}}^{\frac{h}{2}}\rho(z)dz \quad (17)$$

The harmonic solution can be shown as below:

$$u_3^o(x,t)= =\left[C_1\cos(\beta x)+C_2\sin(\beta x)+C_3\cosh(\beta x)+C_4\sinh(\beta x)\right] \times e^{i\omega t} \quad (18)$$

Where C_1, C_2, C_3, C_4 are constants. These constants are unique for a given set of boundary conditions. So four boundary conditions can be listed here with subscripts' C', 'S' and 'F' referring to the clamp, supported and free condition, respectively.

SS: $$u_3^o(x)=\frac{d^2u_3^o(x)}{dx^2}=0\,,\ x = 0, L \quad (19)$$

CS: $$u_3^o(x)=\frac{du_3^o(x)}{dx}=0\,,\ x = 0 \quad (20)$$

$$u_3^o(x)=\frac{d^2u_3^o(x)}{dx^2}=0\,,\ x = L$$

CC: $$u_3^o(x)=\frac{du_3^o(x)}{dx}=0\,,\ x = 0, L \quad (21)$$

CF: $$u_3^o(x)=\frac{du_3^o(x)}{dx}=0\,,\ x = 0 \quad (22)$$

$$\frac{d^2u_3^o(x)}{dx^2}=\frac{d^3u_3^o(x)}{dx^3}=0\,,\ x = 0$$

By using four equations for each case (SS, CS, CC, or CF) related to the above boundary conditions and (18), the natural frequencies can be obtained from

$$\left|[K]-\left(\frac{\beta}{L}\right)^4[M]\right|=0 \quad (23)$$

$$\omega=\left(\frac{\beta}{L}\right)^2\sqrt{\zeta} \quad (24)$$

β can take values β_1, β_2, … β_n corresponding to the number of the wave associated with the n^{th} mode.

Finite element approach

A different way to calculate natural frequencies is given by using the finite element strategy. A beam element with two nodes is considered. Besides, three degrees of freedom are also considered at each node. They are written as below

$$u_1^o(x)=\sum_{i=1}^{2}u_{1i}^o\psi_i \quad (25)$$

$$u_3^o(x)=\sum_{i=1}^{4}d_i\varphi_i \quad (26)$$

where ψ_i are the Lagrange shape functions, φ_i are the Hermite shape functions. Moreover, u_{11}^o and u_{12}^o are the nodal axial displacements, $d_1=u_{31}^o, d_2=\theta_1^o, d_3=u_{32}^o, d_4=\theta_2^o$ are the nodal transverse displacements and rotations. Note that '1' and '2' refer to nodes 1 and 2 as shown in Figure 2.

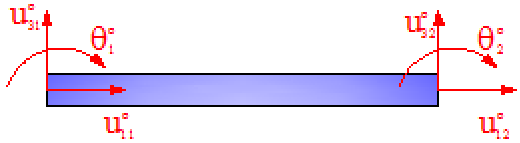

Fig. 2 The classical beam element.

By using the principle of virtual work, the weak forms of the governing equations of motion can be easily obtained. Substituting the expressions for the degree of freedom vector in the weak forms and rearranging, the finite element system of equations can be reached as below

a. Input data
   Geometric data and material properties
b. Calculating constitutive matrix
c. Loop over elements
   Calculating strain matrix
   Calculating element stiffness matrix
   Calculating element mass matrix
d. Assembling the element stiffness and mass matrices in the global coordinate system
e. Applying BCs
f. Solving an equation for free vibration analysis
g. Display natural frequencies.

Results

In this section, with the first example, the first three normalized natural frequencies $\bar{\omega} = \omega L^2 \sqrt{\rho A / EI}$ for the free vibration of an isotropic beam subjected to different sets of boundary conditions using the analytical approach (AS) and finite element approach (FES) are examined. A beam with the following parameters has been considered in the analysis E = 30*e*6, L = 10, ρ = 1. The present results as Table 1, Table 2 and Figure 3 agree with the previous work in [13] by Eltaher *et al.*. With the application of classical beam theory, reliable results can be obtained in both papers.

In the next example, Table 3 shows the first three normalized natural frequencies $\bar{\omega} = \omega / \sqrt{\zeta_o}$ of the exponential functionally graded beam with SS boundary condition and the value ζ of an isotropic beam.

Table 1 The first three normalized natural frequencies for an isotropic beam under various boundary conditions with L/h = 20.

L/h	Method		$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$
20	SS	AS	9.9104	39.8252	89.8749
		FES	9.8861	39.6519	89.7079
		[13]	9.8798	39.6460	89.7046
	CS	AS	15.4504	50.2218	105.4712
		FES	15.4375	50.1935	105.3609
		[13]	15.4368	50.1982	105.3552
	CC	AS	24.5237	62.0483	122.4119
		FES	24.4158	61.9904	122.3367
		[13]	24.4022	61.9872	122.2778

Table 2 The first three normalized natural frequencies for an isotropic beam under various boundary conditions with L/h = 100.

L/h	Method		$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$
100	SS	AS	9.9006	39.6287	89.2124
		FES	9.8819	39.5164	88.9763
		[13]	9.8700	39.4849	88.8595
	CS	AS	15.4325	50.1138	104.9754
		FES	15.4207	50.0762	104.3159
		[13]	15.4189	49.9738	104.2888
	CC	AS	22.8876	62.0019	121.2442
		FES	22.4017	61.8916	121.0417
		[13]	22.3744	61.6847	120.9536

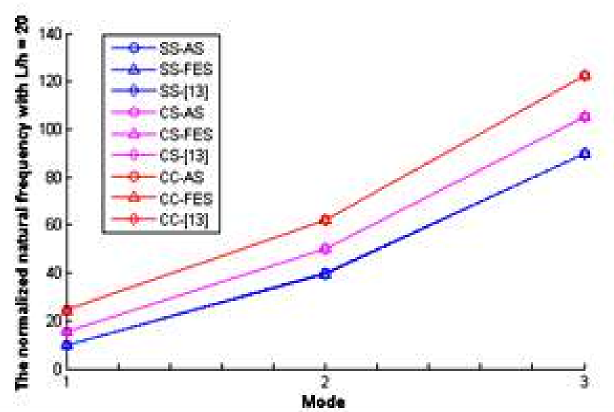

Fig. 3 The comparison of the normalized natural frequencies of an isotropic beam

Table 3 The first three normalized natural frequencies for SS exponential functionally graded beam with L/h = 10.

E_2 / E_1	BCs (SS)	[14]	AS	FES
0.2	$\bar{\omega}_1$	9.270	9.261	9.258
	$\bar{\omega}_2$	37.09	37.112	37.044
	$\bar{\omega}_3$	83.28	83.271	83.269
1	$\bar{\omega}_1$	9.87	9.871	9.828
	$\bar{\omega}_2$	39.48	39.479	39.465
	$\bar{\omega}_3$	88.83	88.830	88.828

Finally, by changing the boundary condition from SS to CF and CC, the first three normalized natural frequencies are also determined and given in Table 4.

The present results as Table 3 and Figure 4 for exponential functionally graded beam also agree with the other work in [14] by authors Liu and Shu.

Table 4 The first three normalized natural frequencies for CF, CC exponential functionally graded beams with L/h = 10.

E_2 / E_1	BCs (CF)	AS	FES
0.2	$\bar{\omega}_1$	3.17	3.13
	$\bar{\omega}_2$	19.91	19.85
	$\bar{\omega}_3$	58.04	57.91
1	$\bar{\omega}_1$	3.38	3.27
	$\bar{\omega}_2$	21.87	21.79
	$\bar{\omega}_3$	60.76	60.52
E_2 / E_1	BCs (CC)	AS	FES
0.2	$\bar{\omega}_1$	20.917	20.643
	$\bar{\omega}_2$	57.123	57.225
	$\bar{\omega}_3$	112.725	112.491
1	$\bar{\omega}_1$	21.894	21.521
	$\bar{\omega}_2$	61.621	61.479
	$\bar{\omega}_3$	120.303	120.182

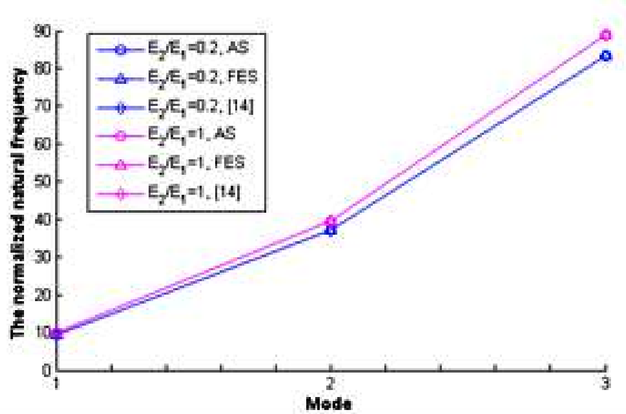

Fig. 4. The comparison of the normalized natural frequencies of SS exponential functionally graded beam.

Conclusion

In this work, the authors present two ways to calculate the natural frequencies of an exponential functionally graded beam under four different types of boundary conditions. The results of this paper are good, agree well with others in references. Although the topic and approach of the paper are not new, the authors' main aim is to reaffirm the applicability of classical beam theory once again to analyze the functionally graded beams with acceptable results. Furthermore, among three kinds of functionally graded materials, exponential material is rarely used for analysis, so this paper's mechanical information may also be helpful to designers for specific purposes.

Acknowledgments

The authors themselves did this study.

References

1. Shen, H.S., *Thermal postbuckling behavior of shear deformable FGM plates with temperature-dependent properties*, International Journal of Mechanical Sciences, 2007. **49**(4): p. 466-478.
2. Simsek, M., *Static analysis of a functionally graded beam under a uniformly distributed load by Ritz method*, International Journal of Engineering and Applied Sciences, 2009. **1**(3): p. 1-11.
3. Ton, H.L.T., *Finite element analysis of functionally graded skew plates in thermal environment based on the new third-order shear deformation theory*, Journal of Applied and Computational Mechanics, 2020. **6**(4): p. 1044-1057.
4. Ton-That, H.L., *Improvement on eight-node quadrilateral element (IQ8) using twice-interpolation strategy for linear elastic fracture mechanics*, Engineering Solid Mechanics, 2020. **8**(4): p. 323-336.
5. Kadoli, R., et al., *Static analysis of functionally graded beams using higher order shear deformation theory*, Applied Mathematical Modelling, 2008. **32**(12): p. 2509–2525.
6. Li, XF, et al., *A higher-order theory for static and dynamic analyses of functionally graded beams*, Archive of Applied Mechanics, 2010. **80**(10): p. 1197–1212.
7. Zhong, Z. and Yu, T., *Analytical solution of a cantilever functionally graded beam*, Composites Science and Technology, 2007. **67**(3): p. 481–488.
8. Hoang-Lan, T.T., *A Combined Strain Element to Functionally Graded Structures in Thermal Environment*, Acta Polytechnica, 2020, **60**(6): p. 528-539.
9. Sankar, B.V., *An elasticity solution for functionally graded beams*, Composites Science and Technology, 2001. **61**: p. 689–696.
10. Khalili, S.M.R., et al., *A mixed Ritz-DQ method for forced vibration of functionally graded beams carrying moving loads*, Composite Structure, 2010. **92**: p. 2497–2511.
11. Mohanty, S.C., et al., *Parametric instability of a functionally graded Timoshenko beam on Winkler's foundation*, Nuclear Engineering and Design, 2011. **241**: p. 2698–2715.
12. Hoang-Lan, T.T., *The Linear and Nonlinear Bending Analyses of Functionally Graded Carbon Nanotube-Reinforced Composite Plates Based on the Novel Four-Node Quadrilateral Element*, European Journal of Computational Mechanics, 2020. **29**(1): p. 139-172.
13. Eltaher, M.A., et al., *Vibration analysis of Euler–Bernoulli nanobeams by using finite element method*, Applied Mathematical Modelling, 2013. **37**: p. 4787-4797.
14. Liu, Y. and Shu, D.W., *Free vibration analysis of exponential functionally graded beams with a single delamination*, Composites: Part B, 2014. **59**: p. 166-172.
15. Filippi, M., et al., *Static analyses of FGM beams by various theories and finite elements*, Composites: Part B, 2015. **72**: p. 1-9.
16. Ziou, H., et al., *Numerical modelling of a Timoshenko FGM beam using the finite element method*, International Journal of Structural Engineering, 2016. **7**(3): p. 239-261.
17. Khan. AA, et al., *Finite Element Modelling for Static and Free Vibration Response of Functionally Graded Beam*,

Latin American Journal of Solids and Structures, 2016. **13**(4): p. 690-714.

18. Madenci, E., *A refined functional and mixed formulation to static analyses of fgm beams*, Structural Engineering and Mechanics, 2019. **69**(4): p. 427-437.
19. Sina, S.A., et al., *An analytical method for free vibration analysis of functionally graded beams*, Materials & Design, 2009. **30**(3): p. 741-747.
20. Ton-That, H.L., *A new C0 third-order shear deformation theory for the nonlinear free vibration analysis of stiffened functionally graded plates*, Facta Universitatis, Series: Mechanical Engineering, 2021, **19**(2): p. 285-305.
21. Simsek, M. and Reddy, J.N., *Bending and vibration of functionally graded microbeams using a new higher order beam theory and the modified couple stress theory*, International Journal of Engineering Science, 2013. **64**: p. 37-53.