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The Application of Homotopy Analysis Method to Non-linear Reaction Diffusion System

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Abstract

In this paper, the homotopy analysis method is applied to non-linear reaction diffusion system of Lotka-Volterra type subject of extensive numerical and analytical studies. The comparisons of the other analytical techniques are presented in tables to show the accuracy of this method. The results show that the homotopy analysis method is more reliable than the other available techniques giving the advantage of the choice of some quantities such as initial guess, auxiliary function, auxiliary parameter that play an important role in the convergence of the series solution.

Keywords: Predator prey interactions; Comparison of perturbation methods; Homotopy analysis method

Introduction

In the recent years, for solving a wide range of mathematical, engineering and physical problems, linear and nonlinear, many more of the numerical methods are used. The homotopy analysis method (HAM), proposed first by Liao in 1992, for solving differential and integral equations, linear and nonlinear, are applied to non-linear reaction diffusion system of Lotka-Volterra type subject of extensive numerical and analytical studies. The proposed method has been applied successfully on many non-linear problems giving promising results [6].

The Lotka-Volterra model is a system which is constructed as the form of the system of first order non-linear differential equations. This system of equations is also called the predator-prey equations. Moreover, the interactions between two animals or species in which one of them performs as a predator that kills and eats up the other species, and the species that it eaten by the predator is called its prey. In this situation, when the growth rate of one population is decreasing and the other is increasing then the populations are said to be in predator-prey conditions.

The Lotka-Volterra predator-prey models were firstly presented by Alfred J. Lotka in his theory of chemical reactions of auto-catalystin 1910 [3].In 1926, Vito Volterra[5], investigated the equations autonomously and prepared a statistical analysis of catching fishes in the Adriatic Sea.VitoVolterra applied these equations to predator prey relations involving a pair of first order autonomous system of equations. Ever since that time, the Lotka-Voltera model has been successfully applied to problems in chemical kinetics, population biology, epidemiology and neural systems.

As given by Volterra if the prey population is and the predator is with respect to time then the Volterra's model is [4]:

$$\dot{x} = x(e - fy), \tag{1}$$

$$\dot{y} = y(gx - h). \tag{2}$$

Where

- represents the population of preys (such as, tiger);
- represents the population of predator (such as, deer);
- and denote the rates of growth of the two populations respectively with respect to time ;
- represents the time;
- are the parameters representing the two species interactions;

So the above pair of equations (1) and (2) is known as the Lotka–Volterra model.

Nonlinear Reaction Diffusion System of Lotka-Volterra Type

Consider the following system of nonlinear Lotka-Volterra type of partial differential equations [4]:

$$u_t = (uu_x)_x + u(a + bu) + h_1 + cv, \tag{3}$$

$$v_t = (vv_x)_x + v(d + ev) + h_2 + fu. \tag{4}$$

The initial conditionsare given as:

$u(x,0) = f(x),\ v(x,0) = g(x).$

Where a, b, e, f, g, k, p, q all are the arbitrary constants with the conditions $ef \neq 0,\quad gk \neq 0$. The given system of equations also contains quadratic nonlinear terms and the equations are coupled.

The General Solution

An exact periodic solution of this system was obtained in[2]:

$$u(x,t) = \left(\varphi_0(t) \pm \left(\frac{2c}{g} + \varphi_0(t)\right)\cos\left(\sqrt{\frac{g}{2}}x \mp |c|t - \beta\right)\right),$$

(5)

$$v(x,t) = \left(\varphi_0(t) + \frac{4c}{g} \pm \frac{|c|}{c}\left(\varphi_0(t) + \frac{2c}{g}\right)\sin\left(\sqrt{\frac{g}{2}}x \mp |c|t - \beta\right)\right),$$

(6)

$$\varphi_0(t)=\frac{1}{3g}\left(\begin{array}{l}\left(\frac{2}{s-t}-6c\right), a=3c\\ \left(|3c-a|\tanh\left(\frac{|3c-a|}{2}(s-t)\right)-a-3c\right),\\ a\neq 3c\end{array}\right).$$

(7)

Here b,c,e,h_1,f,h_2,β,s all are the arbitrary constants with the condition $b=e=g>0,$

$$d=a-6c,\ f=-c, h_1=\frac{(2ac-6c^2)}{g}, h_2=h_1+\frac{4c}{g}(3c-a).$$

Lysis of the Problem by Using Homotopy Analysis Method (Ham)

The approximate solution of the nonlinear reaction diffusion system of Lotka-Volterratypeis established here by HAM, first by choosing the linear operator as $L=\frac{\partial}{\partial t}$, and as its inverse will be defined as $L^{-1}=\int_0^t(.)dt.$

Thus as stated by HAM, we make the construction of a homotopy and extract the zero order deformation equation, and the higher order deformation equation as,

$$(1-p)L\left[\xi(x,t;p)-u_0\right]=ph\mathrm{H}(x,t)\mathrm{N}\left[\xi(x,t;p)\right]. \quad (8)$$

For the given system of equations becomes, we may have,

$$L\left[\psi_1(x,t;p)-u_0\right]-pL\left[\psi_1(x,t;p)-u_0\right]=ph\mathrm{H}(x,t)\mathrm{N}\left[\psi_1(x,t;p)\right],$$

(9)

$$L\left[\psi_2(x,t;p)-v_0\right]-pL\left[\psi_2(x,t;p)-v_0\right]=ph\mathrm{H}(x,t)\mathrm{N}\left[\psi_2(x,t;p)\right].$$

(10)

By using $p=0$ and $p=1$ in equation (9) and (10) respectively, then we have $\psi_1(x,t;p)\big|_{p=0}=u_0,$ and $\psi_2(x,t;p)\big|_{p=0}=v_0.$ they becomes the initial approximations for the given problem, whereas $\psi_1(x,t;p)\big|_{p=1}=u(x,t),$ and $\psi_2(x,t;p)\big|_{p=1}=v(x,t).$ are the exact solutions. Then the solution of the given system of equations takes the following form,

$$\psi_1(x,t)=u_0(x,t)+\sum_{n=1}^{\infty}p^n u_n(x,t), \quad (11)$$

and $$\psi_2(x,t)=v_0(x,t)+\sum_{n=1}^{\infty}p^n v_n(x,t), \quad (12)$$

Where

$$u_n(x,t)=\frac{1}{n!}\frac{\partial^n\psi_1(x,t;p)}{\partial p^n}, \quad v_n=\frac{1}{n!}\frac{\partial^n\psi_2(x,t;p)}{\partial p^n},$$

are calculated at exist for , and also converges at . Then the solution of the system of equations under study become, are calculated at $p=0,$ exist for $n\geq 1$, and also converges at $p=1$. Then the solution of the system of equations under study become,

$$u(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+..., \quad (13)$$

$$v(x,t)=v_0(x,t)+v_1(x,t)+v_2(x,t)+..., \quad (14)$$

Now according to the Homotopy Analysis Method (HAM), the higher order deformation equation for the given system of equations becomes as,

$$L\left[u_n\right]=\lambda_n L\left[u_{n-1}\right]+hH(x,t)R_n(\overrightarrow{u_{n-1}}), \quad (15)$$

$$L\left[v_n\right]=-\lambda_n L\left[v_{n-1}\right]+hH(x,t)R_n(\overrightarrow{v_{n-1}}). \quad (16)$$

Applying the inverse operator, we may extract the nth order deformation equations solutions as follows,

$$u_n=\lambda_n u_{n-1}+h\int_0^t H.R_n(\overrightarrow{u_{n-1}})dt, \qquad v_n=\lambda_n v_{n-1}+h\int_0^t H.R_n(\overrightarrow{v_{n-1}})dt.$$

(17)

Now for we get the following sets of equations:

$$\begin{cases}u_1=h\int_0^t H.R_1(\overrightarrow{u_0})dt,\\ v_1=h\int_0^t H.R_1(\overrightarrow{v_0})dt.\end{cases} \quad (18)$$

$$\begin{cases}u_2=u_1+h\int_0^t H.R_2(\overrightarrow{u_1})dt,\\ v_2=v_1+h\int_0^t H.R_2(\overrightarrow{v_1})dt.\end{cases} \quad (19)$$

$$\begin{cases}u_3=u_2+h\int_0^t H.R_3(\overrightarrow{u_2})dt,\\ v_3=v_2+h\int_0^t H.R_3(\overrightarrow{v_2})dt.\end{cases} \quad (20)$$

$$\begin{cases}u_4=u_3+h\int_0^t H.R_4(\overrightarrow{u_3})dt,\\ v_4=v_3+h\int_0^t H.R_4(\overrightarrow{v_3})dt.\end{cases} \quad (21)$$

$$\begin{cases} u_5 = u_4 + h\int_0^t H(x,t)R_5(\overrightarrow{u_4})dt, \\ v_5 = v_4 + h\int_0^t H(x,t)R_5(\overrightarrow{v_4})dt. \end{cases} \quad (22)$$

and so on. Since

$$R_n(\vec{u}_{n-1}) = \left.\frac{\partial^{n-1}\left[N\left[u(x,t;p)\right]\right]}{(n-1)!\partial p^{n-1}}\right|_{p=0},$$
$$R_n(\vec{v}_{n-1}) = \left.\frac{\partial^{n-1}\left[N\left[v(x,t;p)\right]\right]}{(n-1)!\partial p^{n-1}}\right|_{p=0} \quad (23)$$

Now putting in (23), we havethe following forms of the above quantities,

$$R_1(\vec{u}_0) = u_{0t} - u_{0x}{}^2 - u_0u_{0xx} - au_0 - bu_0{}^2 - cv_0 - h_1. \quad (24)$$

In the same manner we get,

$$R_1(\vec{v}_0) = v_{0t} - v_{0x}{}^2 - v_0v_{0xx} - dv_0 - ev_0{}^2 - fu_0 - h_2. \quad (25)$$

Now putting $n = 2,$ in (23), we get

$$R_2(\vec{u}_1) = u_{1t} - 2u_{0x}u_{1x} - u_1u_{0xx} - u_{1xx}u_0 - au_1 - 2bu_0u_1 - cv_1. \quad (26)$$

Similarly

$$R_2(\vec{v}_1) = v_{1t} - 2v_{0x}v_{1x} - v_1v_{0xx} - v_{1xx}v_0 - dv_1 - 2ev_0v_1 - fu_1. \quad (27)$$

For $n = 3,$ (23) becomes,

$$R_3(\vec{u}_2) = u_{2t} - 2u_{0x}u_{2x} - u_{0xx}u_2 - u_{1x}{}^2 - u_1u_{1xx} - u_{2xx}u_{0x} - au_2 - 2bu_0u_2 - bu_1{}^2 - cv_2. \quad (28)$$

Similarly

$$R_3(\vec{v}_2) = v_{2t} - 2v_{0x}v_{2x} - v_{0xx}v_2 - v_{1x}{}^2 - v_1v_{1xx} - v_{2xx}v_{0x} - dv_2 - 2ev_0v_2 - ev_1{}^2 - fu_2. \quad (29)$$

For $n = 4,$ (23) becomes,

$$R_4(\vec{u}_3) = u_{3t} - 2u_{0x}u_{3x} - 2u_{1x}u_{2x} - u_{0xx}u_3 - u_2u_{1xx} - u_{2xx}u_1 - u_{3xx}u_0 - au_3 - 2b(u_0u_3 + u_1u_2) - cv_3. \quad (30)$$

In the same manner we get,

$$R_4(\vec{v}_3) = v_{3t} - 2v_{0x}v_{3x} - 2v_{1x}v_{2x} - v_{0xx}v_3 - v_2v_{1xx} - v_{2xx}v_1 - v_{3xx}v_0 - dv_3 - 2e(v_0v_3 + v_1v_2) - fu_3. \quad (31)$$

Putting $n = 5,$ (23) becomes:

$$\begin{aligned} R_5(\vec{u}_4) = {} & u_{3t} - 2u_{0x}u_{4x} - 2u_{1x}u_{3x} - u_{0xx}u_4 - u_2u_{1xx} - u_{4xx}u_0 - u_{1xx}u_3 - u_1u_{3xx} \\ & - u_2u_{2xx} - (u_{2x})^2 - au_4 - \mathrm{b}u_2{}^2 - 2b(u_0u_4 + u_1u_3) - cv_4, \end{aligned} \quad (32)$$

Similarly we get the second component,

$$\begin{aligned} R_5(\vec{v}_4) = {} & v_{3t} - 2v_{0x}v_{4x} - 2v_{1x}v_{3x} - v_{0xx}v_4 - v_2v_{1xx} - v_{4xx}v_0 - v_{1xx}v_3 - v_1v_{3xx} \\ & - v_2v_{2xx} - (v_{2x})^2 - dv_4 - \mathrm{e}(v_2)^2 - 2e(v_0v_4 + v_1v_3) - fu_4. \end{aligned} \quad (33)$$

Using all these calculated values in (18),(19),(20),(21) and (22) respectively,
We get,

$$\begin{cases} u_1 = h\int_0^t H(x,t)(u_{0t} - u_{0x}{}^2 - u_0u_{0xx} - au_0 - bu_0{}^2 - cv_0)dt, \\ v_1 = h\int_0^t H(x,t)(v_{0t} - v_{0x}{}^2 - v_0v_{0xx} - dv_0 - ev_0{}^2 - fu_0)dt. \end{cases} \quad (34)$$

$$\begin{cases} u_2 = u_1 + h\int_0^t H(x,t)(u_{1t} - 2u_{0x}u_{1x} - u_1u_{0xx} - u_{1xx}u_0 - au_1 - 2bu_0u_1 - cv_1)dt \\ v_2 = v_1 + h\int_0^1 H(x,t)(v_{1t} - 2v_{0x}v_{1x} - v_1v_{0xx} - v_{1xx}v_0 - dv_1 - 2ev_0v_1 - fu_1)dt. \end{cases} \quad (35)$$

$$\begin{cases} u_3 = u_2 + h\int_0^t H(x,t)(u_{2t} - 2u_{0x}u_{2x} - u_{0xx}u_2 - {u_{1x}}^2 - u_1u_{1xx} - u_{2xx}u_{0x} - \ au_2 - 2bu_0u_2 - b{u_1}^2 - cv_2)dt, \\ v_3 = v_2 + h\int_0^t H(x,t)(v_{2t} - 2v_{0x}v_{2x} - v_{0xx}v_2 - {v_{1x}}^2 - v_1v_{1xx} - v_{2xx}v_{0x} - dv_2 - 2ev_0v_2 - e{v_1}^2 - fu_2)dt. \end{cases} \tag{36}$$

$$\begin{cases} u_4 = u_3 + h\int_0^t H(x,t)(u_{3t} - 2u_{0x}u_{3x} - 2u_{1x}u_{2x} \\ \quad - u_{0xx}u_3 - u_2u_{1xx} - u_{2xx}u_1 - u_{3xx}u_0 \\ \quad - au_3 - 2b(u_0u_3 + u_1u_2) - cv_3)dt, \\ v_4 = v_3 + h\int_0^t H(x,t)(v_{3t} - 2v_{0x}v_{3x} - 2v_{1x}v_{2x} \\ \quad - v_{0xx}v_3 - v_2v_{1xx} - v_{2xx}v_1 - v_{3xx}v_0 \\ \quad - dv_3 - 2e(v_0v_3 + v_1v_2) - fu_3)dt. \end{cases} \tag{37}$$

$$\begin{cases} u_5 = u_4 + h\int_0^t H(x,t)(u_{4t} - 2u_{0x}u_{4x} - 2u_{1x}u_{3x} - u_{0xx}u_4 - u_0u_{4xx} - u_{4xx}u_0 - u_{1xx}u_3 \\ \quad - u_1u_{3xx} - \ u_2u_{2xx} - (u_{2x})^2 - au_4 - \ \mathrm{b}{u_2}^2 - 2b(u_0u_4 + u_1u_3) - cv_4)dt, \\ v_5 = v_4 + h\int_0^t H(x,t)(v_{4t} - 2v_{0x}v_{4x} - 2v_{1x}v_{3x} - v_{0xx}v_4 - v_0v_{4xx} - v_{4xx}v_0 - v_{1xx}v_3 \\ \quad - v_1v_{3xx} - \mathrm{v}_2v_{2xx} - (v_{2x})^2 - dv_4 - e(v_2)^2 - 2e(v_0v_4 + v_1v_3) - fu_4)dt. \end{cases} \tag{38}$$

and so on.

For the purpose of the solution of (5) and (6), we consider the two possible cases here [1].

Choice of Base Functions

We observe that the equations (16) and (17) are in generalized form. One can use different base functions to express the solution of a nonlinear problem in order to start the analysis of the problem for different cases of interest. In the famous book by Liao, (Beyond perturbation: An Introduction to Homotopy Analysis Method), he gave a detailed idea of choosing different base functions, the range value of the auxiliary parameter, the linear operator and the initial guess. The solution expressions discussed by him include the polynomial functions, fractional functions, exponential functions etc.

It has been shown by him that if we choose the value of the auxiliary parameter as -1, the value of the auxiliary function as 1, then the solution expressions are same as provided by homotopy perturbation method. That is, HAM contains HPM logically. We use the same strategy here, but the difference remains that of the different initial approximation.

Here we discuss two cases to present our analysis;

Case 1: When $a = 3c.$ Here $h_1 = h_2 = 0.$ We start with the initial approximations as,

$$u_0 = \left(\frac{1}{3}\frac{\left(\frac{2}{s} - 6c\right)}{g}\right) + \left(\frac{1}{3}\frac{\left(\frac{2}{s} - 6c\right)}{g} + \frac{2c}{g}\right)\left(\cos\left(-\frac{1}{2}x\sqrt{2g} + \beta\right)\right),$$

(39)

$$v_0 = \left(\frac{1}{3}\frac{\left(\frac{2}{s} - 6c\right)}{g}\right) + \frac{4c}{g} + \left(\frac{1}{3}\frac{\left(\frac{2}{s} - 6c\right)}{g} + \frac{2c}{g}\right)\left(\sin\left(-\frac{1}{2}x\sqrt{2g} + \beta\right)\right).$$

(40)

Now we have to calculate u_1, from equation (34) we have,

$$u_1 = h(-a\left(\frac{2}{3gs} - \frac{2}{g} + \frac{2}{3}\frac{\cos\left(\frac{1}{2}\sqrt{2}\sqrt{g}x - \beta\right)}{gs}\right)t - b\left(\frac{2}{3gs} - \frac{2}{g} + \frac{2}{3}\frac{\cos\left(\frac{1}{2}\sqrt{2}\sqrt{g}x - \beta\right)}{gs}\right)^2 t$$
$$-c\left(\frac{2}{3gs} + \frac{2}{g} + \frac{2}{3}\frac{\sin\left(\frac{1}{2}\sqrt{2}\sqrt{g}x - \beta\right)}{gs}\right)t - h_1t + \frac{\left(\frac{2}{3gs} - \frac{2}{g} + \frac{2}{3}\frac{\cos\left(\frac{1}{2}\sqrt{2}\sqrt{g}x - \beta\right)}{gs}\right)}{3s}$$
$$\cos\left(\frac{1}{2}\sqrt{2}\sqrt{g}x - \beta\right)t\ \frac{2}{9}\frac{\sin\left(\frac{1}{2}\sqrt{2}\sqrt{g}x - \beta\right)^2 t}{gs^2}\Bigg). \tag{41}$$

In the same manner we get v_1 as:

$$v_1 = h\left(-d\left(\frac{2}{3gs}+\frac{2}{g}+\frac{2}{3}\frac{\sin\left(\frac{1}{2}\sqrt{2}\sqrt{g}x-\beta\right)}{gs}\right)t - e\left(\frac{2}{3gs}+\frac{2}{g}+\frac{2}{3}\frac{\sin\left(\frac{1}{2}\sqrt{2}\sqrt{g}x-\beta\right)}{gs}\right)^2 t\right.$$
$$-f\left(\frac{2}{3gs}-\frac{2}{g}+\frac{2}{3}\frac{\cos\left(\frac{1}{2}\sqrt{2}\sqrt{g}x-\beta\right)}{gs}\right)t - h_2 t - \frac{2}{9}\left(\frac{\cos\left(\frac{1}{2}\sqrt{2}\sqrt{g}x-\beta\right)^2 t}{gs^2}\right)$$
$$\left.+\frac{1}{3s}\left(\frac{2}{3gs}+\frac{2}{g}+\frac{2}{3}\frac{\sin\left(\frac{1}{2}\sqrt{2}\sqrt{g}x-\beta\right)}{gs}\right)\sin\left(\frac{1}{2}\sqrt{2}\sqrt{g}x-\beta\right)t\right). \quad (42)$$

For u_2, from (35) we have, and putting $a=3, b=2, c=1, d=-3, e=2, f=-1,$ in (35), we get:

$$u_2 = h(-3\left(-\frac{11}{12}+\frac{1}{12}\cos(x-3)\right)t - 4\left(-\frac{11}{12}+\frac{1}{12}\cos(x-3)\right)^2 t - \left(\frac{13}{12}+\frac{1}{12}\sin(x-3)\right)t$$
$$+\frac{1}{6}\left(-\frac{11}{12}+\frac{1}{12}\cos(x-3)\right)\cos(x-3)t - \frac{1}{72}\sin(x-3)^2 t + (-\frac{22}{3}+\frac{2}{3}\cos(x-3)+$$
$$6\left(-\frac{11}{12}+\frac{1}{12}\cos(x-3)\right)^2 - \frac{1}{4}\left(-\frac{11}{12}+\frac{1}{12}\cos(x-3)\right)\cos(x-3)+\frac{1}{48}\sin(x-3)^2$$
$$-4\left(-\frac{11}{12}+\frac{1}{12}\cos(x-3)\right) + (\frac{5}{3}-\frac{1}{4}\cos(x-3)-2\left(-\frac{11}{12}+\frac{1}{12}\cos(x-3)\right)^2$$
$$-\frac{1}{12}\sin(x-3)+\frac{1}{12}\left(-\frac{11}{12}+\frac{1}{12}\cos(x-3)\right)\cos(x-3)-\frac{1}{144}\sin(x-3)^2)$$
$$2\left(\frac{13}{12}+\frac{1}{12}\sin(x-3)\right)^2 - \frac{1}{12}\left(\frac{13}{12}+\frac{1}{12}\sin(x-3)\right)\sin(x-3)+\frac{1}{144}\cos(x-3)^2 -$$
$$\left(\frac{1}{4}\cos(x-3)+\frac{1}{4}\left(-\frac{11}{12}+\frac{1}{12}\cos(x-3)\right)\cos(x-3)+\frac{1}{12}\sin(x-3)-\frac{1}{48}\cos(x-3)^2\right)$$
$$\left(-\frac{11}{12}+\frac{1}{12}\cos(x-3)\right)+\frac{1}{6}\sin(x-3)(\frac{1}{4}\sin(x-3)+\frac{1}{4}\left(-\frac{11}{12}+\frac{1}{12}\cos(x-3)\right)$$
$$\sin(x-3)-\frac{1}{12}\cos(x-3)-\frac{1}{48}\sin(x-3)\cos(x-3))$$
$$+\frac{1}{12}\cos(x-3)(\frac{5}{3}-\frac{1}{4}\cos(x-3)-2\left(-\frac{11}{12}+\frac{1}{12}\cos(x-3)\right)^2-\frac{1}{12}\sin(x-3)$$
$$+\frac{1}{12}\left(-\frac{11}{12}+\frac{1}{12}\cos(x-3)\right)\cos(x-3)-\frac{1}{144}\sin(x-3)^2))$$
$$t^2+\frac{5}{3}t-\frac{1}{4}\cos(x-3)t-\frac{1}{12}\sin(x-3)t). \quad (43)$$

Now for the second component v_2, from (35) we have,

$$v_2 = h(10\left(\frac{13}{12}+\frac{1}{12}\sin(x-3)\right)t-8\left(\frac{13}{12}+\frac{1}{12}\sin(x-3)\right)^2 t-2\left(-\frac{11}{12}+\frac{1}{12}\cos(x-3)\right)t+\frac{1}{3}\left(\frac{13}{12}+\frac{1}{12}\sin(x-3)\right)\sin(x-3)t$$
$$-\frac{1}{36}\cos(x-3)^2 t-4\left(\frac{13}{12}+\frac{1}{12}\sin(x-3)\right)(3\left(\frac{13}{12}+\frac{1}{12}\sin(x-3)\right)t-2\left(\frac{13}{12}+\frac{1}{12}\sin(x-3)\right)^2 t+\left(-\frac{11}{12}+\frac{1}{12}\cos(x-3)\right)t+$$
$$\frac{1}{12}\left(\frac{13}{12}+\frac{1}{12}\sin(x-3)\right)\sin(x-3)t-\frac{1}{144}\cos(x-3)^2 t)-4\left(-\frac{11}{12}+\frac{1}{12}\cos(x-3)\right)^2 +\frac{1}{6}\left(-\frac{11}{12}+\frac{1}{12}\cos(x-3)\right)\cos(x-3)t$$
$$-\frac{1}{72}\sin(x-3)^2 t+\frac{7}{3}+\frac{1}{4}\sin(x-3)-2\left(\frac{13}{12}+\frac{1}{12}\sin(x-3)\right)^2+\frac{1}{12}\cos(x-3)+\frac{1}{12}\left(\frac{13}{12}+\frac{1}{12}\sin(x-3)\right)\sin(x-3)-$$
$$\frac{1}{12}\cos(x-3)t-\frac{1}{48}\sin(x-3)^2 t\left(\frac{13}{12}+\frac{1}{12}\sin(x-3)\right)-\frac{1}{6}\cos(x-3)\ (\frac{1}{4}\cos(x-3)t-\frac{1}{4}\left(\frac{13}{12}+\frac{1}{12}\sin(x-3)\right)t\cos(x-3)$$
$$-\frac{1}{12}\sin(x-3)t+\frac{1}{48}\cos(x-3)\sin(x-3)t)+\frac{1}{12}\sin(x-3)(3\left(\frac{13}{12}+\frac{1}{12}\sin(x-3)\right)t-2\left(\frac{13}{12}+\frac{1}{12}\sin(x-3)\right)^2 t$$
$$+\left(-\frac{11}{12}+\frac{1}{12}\cos(x-3)\right)t+\frac{1}{12}\left(\frac{13}{12}+\frac{1}{12}\sin(x-3)\right)\sin(x-3)t-\frac{1}{144}\cos(x-3)^2 t)). \quad (44)$$
$$\vdots$$

And so on.

We calculate the approximate6-terms solution series given by HAM as;

$$\psi_1(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+u_4(x,t)+u_5(x,t), \quad (45)$$
$$\psi_2(x,t)=v_0(x,t)+v_1(x,t)+v_2(x,t)+v_3(x,t)+v_4(x,t)+v_5(x,t). \quad (46)$$

Table 1 and Table 2shows the comparison of numerical solutions by HAM with the exact solutions of equation (3) and (4) for u and v, when $h=-1/2,\ and\ H(x,t)=1.$ Where $a=3, b=2, c=1, d=-3, e=2, f=-1,$ $h_1=0, h_2=0, s=4,$ and $t=.6,$ and for different values of $x.$ Case 2: For $a\neq 3c.$ Then

$$\theta_0(t)=\frac{1}{3g}\left(|3c-a|\tanh\left(\frac{|3c-a|}{2}(s-t)\right)-a-3c\right),$$

Using $a=3, b=3, c=1.2, d=-3, e=2, f=-1.2,\ h_1=1, h_2=1.96, \beta=3,\ and\ s=4.$

Taking the initial approximations

$$\tilde{u}_0=-0.6975+0.1024\cos\left(-3+\frac{1}{2}\sqrt{6}x\right), \quad (47)$$
$$\tilde{v}_0=0.9024-0.1024\sin\left(3-\frac{1}{2}\sqrt{6}x\right). \quad (48)$$

The other components calculated by Maple are given by:

$$\tilde{u}_1=h(-0.01572864000\sin(-3.+1.224744782x)^2 t+0.1536000000$$
$$\left(-0.6975000000+0.1024000000\cos(-3.+1.224744782x)\right)$$
$$\cos(-3.+1.224744782x)t+0.009620000000t-0.3072000000$$
$$\cos(-3.+1.224744782x)t-3.(-0.6975000000+0.1024000000$$
$$\cos(-3.+1.224744782x))^2 t-0.1228800000\sin(-3.+1.224744782x)t). \quad (49)$$
$$\tilde{v}_1=h(-0.01572864000\cos(-3.+1.224744782x)^2 t+0.1536000000$$
$$\left(0.9024000000+0.1024000000\sin(-3.+1.224744782x)\right)\sin(-3.+1.224744782x)t$$
$$+1.953080000t+0.4300800000\sin(-3.+1.224744782x)t$$
$$-3.\left(0.9024000000+0.1024000000\sin(-3.+1.224744782x)\right)^2 t$$
$$+0.1228800000\cos(-3.+1.224744782x)t). \quad (50)$$

Table 1: For $\psi_1(x,t)$.

x_j	Exact Solution	HAM Solution	Absolute Error
-8	−0.8462461147	-0.8462461328	1.81×10^{-8}
-6	−0.9984988094	-0.9984988237	1.43×10^{-8}
-4	−0.8773274664	-0.8773274192	4.68×10^{-8}
-2	−0.8259249139	-0.8259249615	4.76×10^{-8}
0	−0.9898782761	-0.9898782401	3.60×10^{-8}
2	−0.9048234826	-0.9048234712	1.14×10^{-8}
4	−0.8116606868	-0.8116606196	6.72×10^{-8}
6	−0.9742542858	-0.9742542936	7.8×10^{-9}
8	−0.9320914578	-0.9329914127	4.51×10^{-8}

Table 2: For $\psi_2(x,t)$.

x_j	Exact Solution	HAM Solution	Absolute Error
-8	1.178708686	1.178708640	4.6×10^{-8}
-6	1.115130077	1.115130011	6.6×10^{-8}
-4	1.003145131	1.003145109	2.2×10^{-8}
-2	1.159928102	1.159928131	2.9×10^{-8}
0	1.141423573	1.141423501	7.2×10^{-8}
2	1.000041804	1.000041824	2.0×10^{-8}
4	1.136217485	1.136217419	6.6×10^{-8}
6	1.164261096	1.164261037	5.9×10^{-8}
8	1.004744895	1.004744868	2.7×10^{-8}

Table 3: For $\tilde{\psi}_1(x,t)$.

$t = 0.001,$ and different value of $x,$ when $h = -1/2, \; and \; H(x,t) = 1.$

x_j	Exact Solution	HAM Solution	Absolute Error
-8	−0.5978337669	-0.5978203209	1.34×10^{-5}
-6	−0.7592348358	-0.7592199650	1.49×10^{-5}
-4	−0.7021062252	-0.7021443070	3.81×10^{-5}
-2	−0.6286724457	-0.6286640010	8.44×10^{-5}
0	−0.7988752317	-0.7988092213	6.60×10^{-5}
2	−0.6102288109	-0.6102466101	1.78×10^{-5}
4	−0.7305059455	-0.7305309980	2.51×10^{-5}
6	−0.7339482560	-0.7339119033	3.64×10^{-5}
8	−0.6083706122	-0.6083642019	6.41×10^{-6}

$$\tilde{u}_2 = h\Big(-0.031457(\sin(-3.+1.2247x)^2 t + 0.30720(-0.69750+0.10240\cos(-3.+1.2247x))$$
$$\cos(-3.+1.2247x)t + 0.019240t - 0.61440\cos(-3.+1.2247x)t$$
$$-6.(-0.69750+0.10240\cos(-3.+1.2247x))^2 t - 0.24576\sin(-3.+1.2247x)t +$$
$$0.50000(-6.(-0.015729(sin(-3.+1.2247x))^2 + 0.15360$$
$$(-0.69750+0.10240cos(-3.+1.2247x)) + 0.50000\ (-6.(-0.015729\sin(-3.+1.2247x)^2$$
$$+0.15360(-0.69750+0.10240\cos(-3.+1.2247x))\cos(-3.+1.2247x)$$
$$+0.0096200 - 0.30720\cos(-3.+1.2247x) - 3.(-0.69750+0.10240\cos(-3.+1.2247x))^2$$
$$-0.12288\sin(-3.+1.2247x))(-0.69750+0.10240\cos(-3.+1.2247x))$$
$$+0.047186\sin(-3.+1.2247x)^2 - 0.46080(-0.69750+0.10240\cos(-3.+1.2247x))$$
$$\cos(-3.+1.2247x) - 2.3726 + 0.77414\ \cos(-3.+1.2247x)$$
$$+9.(-0.69750+0.10240\cos(-3.+1.2247x))^2 - 0.14746\ \sin(-3.+1.2247x)$$
$$+0.018874\cos(-3.+1.2247x)^2 - 0.18432(0.90240+0.10240\sin(-3.+1.2247x))$$
$$\sin(-3.+1.2247x) + 3.6000(0.90240+0.10240\sin(-3.+1.2247x))^2$$
$$+0.15360(-0.015729\ \sin(-3.+1.2247x)^2 + 0.15360(-0.69750+0.10240$$
$$\cos(-3.+1.2247x)) + 0.0096200 - 0.30720\ \cos(-3.+1.2247x)$$
$$-3.(-0.69750+0.10240\cos(-3.+1.2247x))^2$$
$$-0.12288\ \sin(-3.+1.2247x))\cos(-3.+1.2247x) + 0.25083$$
$$(-0.057791\sin(-3.+1.2247x)(+0.56436(-0.69750+0.10240\cos(-3.+1.2247x))$$
$$\sin(-3.+1.2247x) + 0.37624\ \sin(-3.+1.2247x) - 0.15050\cos(-3.+1.2247x))$$
$$\sin(-3.+1.2247x) - 1.\ (-0.070779\cos(-3.+1.2247x)^2 - 1.000010^{-11}$$
$$\sin(-3.+1.2247x)^2 + 0.69120(-0.69750+0.10240\cos(-3.+1.2247x))$$
$$\cos(-3.+1.2247x) + 0.46080\cos(-3.+1.2247x) + 0.18432\sin(-3.+1.2247x))$$
$$(-0.69750+0.10240\ \cos(-3.+1.2247x)))t^2\Big). \quad (51)$$

Table 4: For $\tilde{\psi}_2(x,t)$.

x_j	Exact Solution	HAM Solution	Absolute Error
-8	0.8788967666	**0.8788422909**	5.45×10^{-5}
-6	0.9840980419	**0.9840164022**	8.16×10^{-5}
-4	0.8001036526	**0.8001409880**	3.73×10^{-5}
-2	0.9782190462	**0.9782664041**	4.74×10^{-5}
0	0.8879493112	**0.8879001912**	4.91×10^{-5}
2	0.8488322900	**0.8488550406**	2.28×10^{-5}
4	0.9993348625	**0.9993205187**	1.43×10^{-5}
6	0.8067062979	**0.8067119025**	5.60×10^{-5}
8	0.9528153967	**0.9528045663**	$1.08\times10^{=5}$

$$\tilde{v}_2 = h\Big(-0.031457\cos\left(-3.+1.2247x\right)^2 t + 0.30720\left(0.90240+0.10240\sin\left(-3.+1.2247x\right)\right)$$
$$\sin\left(-3.+1.2247x\right)t + 3.9062t + 0.86016\sin\left(-3.+1.2247x\right)t -$$
$$-6.\left(0.90240+0.10240\sin\left(-3.+1.2247x\right)\right)^2 t + 0.24576\cos\left(-3.+1.2247x\right)t$$
$$+0.50000(-6.\,(-0.015729\cos\left(-3.+1.2247x\right)^2$$
$$+0.15360\left(0.90240+0.10240\sin\left(-3.+1.2247x\right)\right)\sin\left(-3.+1.2247x\right)$$
$$+1.9531+0.43008\sin\left(-3.+1.2247x\right)-3.\left(0.90240+0.10240\sin\left(-3.+1.2247x\right)\right)^2$$
$$+0.12288\cos\left(-3.+1.2247x\right))\left(0.90240+0.10240\sin\left(-3.+1.2247x\right)\right)$$
$$-0.066060\cos\left(-3.+1.2247x\right)^2+0.64512\left(0.90240+0.10240\sin\left(-3.+1.2247x\right)\right)$$
$$\sin\left(-3.+1.2247x\right)+8.2145+1.6589\sin\left(-3.+1.2247x\right)-12.600$$
$$\left(0.90240+0.10240\sin\left(-3.+1.2247x\right)\right)^2+0.14746\cos\left(-3.+1.2247x\right)$$
$$-0.018874\sin\left(-3.+1.2247x\right)^2+0.18432\left(-0.69750+0.10240\cos\left(-3.+1.2247x\right)\right)$$
$$\cos\left(-3.+1.2247x\right)-3.6000\left(-0.69750+0.10240\cos\left(-3.+1.2247x\right)\right)^2$$
$$+0.15360\,(-0.015729\cos\left(-3.+1.2247x\right)^2+0.15360$$
$$\left(0.90240+0.10240\sin\left(-3.+1.2247x\right)\right)\sin\left(-3.+1.2247x\right)+1.9531$$
$$+0.43008\sin\left(-3.+1.2247x\right)-3.\left(0.90240+0.10240\sin\left(-3.+1.2247x\right)\right)^2$$
$$+0.12288\cos\left(-3.+1.2247x\right))\sin\left(-3.+1.2247x\right)-0.25083$$
$$(0.057791\sin\left(-3.+1.2247x\right)\cos\left(-3.+1.2247x\right)-0.56436$$
$$\left(0.90240+0.10240\sin\left(-3.+1.2247x\right)\right)\cos\left(-3.+1.2247x\right)+0.52674$$
$$\cos\left(-3.+1.2247x\right)-0.15050\sin\left(-3.+1.2247x\right))\cos\left(-3.+1.2247x\right)-1.$$
$$(-1.000010^{-11}\cos\left(-3.+1.2247x\right)^2-0.070779\sin\left(-3.+1.2247x\right)^2$$
$$+0.69120\left(0.90240+0.10240\sin\left(-3.+1.2247x\right)\right)\sin\left(-3.+1.2247x\right)$$
$$-0.64512\sin\left(-3.+1.2247x\right)-0.18432\cos\left(-3.+1.2247x\right))$$
$$\left(0.90240+0.10240\sin\left(-3.+1.2247x\right)\right))t^2\Big). \tag{52}$$
$$\vdots$$

And so on.

Following the same procedure as in case 1, we finally obtain the 6-terms approximations solution given by

$$\tilde{\psi}_1(x,t) = \tilde{u}_0(x,t)+\tilde{u}_1(x,t)+\tilde{u}_2(x,t)+\tilde{u}_3(x,t)+\tilde{u}_4(x,t)+\tilde{u}_5(x,t), \quad (53)$$

$$\tilde{\psi}_2(x,t) = \tilde{v}_0(x,t)+\tilde{v}_1(x,t)+\tilde{v}_2(x,t)+\tilde{v}_3(x,t)+\tilde{v}_4(x,t)+\tilde{v}_5(x,t). \quad (54)$$

Table 3 and Table 4 list the approximate solutions by the HAM and the absolute errors between the exact solutions and the numerical solutions for $u(x,t)$ and $v(x,t)$ at

Comparison of the numerical results obtained by HAM and ADM presented in the following tables:

Case 1:

Table A:For $u(x,t)$.

For $v(x,t)$.

x_j	$\lvert u_{exact} - u_{HAM}\rvert$	$\lvert u_{exact} - u_{ADM}\rvert$	$\lvert v_{exact} - v_{HAM}\rvert$	$\lvert v_{exact} - v_{ADM}\rvert$
-8	1.81×10^{-8}	2.65×10^{-06}	4.6×10^{-8}	1.11×10^{-06}
-6	1.43×10^{-8}	2.50×10^{-06}	6.6×10^{-8}	3.44×10^{-06}
-4	4.68×10^{-8}	1.57×10^{-06}	2.2×10^{-8}	1.41×10^{-06}
-2	4.76×10^{-8}	1.97×10^{-06}	2.9×10^{-8}	1.45×10^{-06}
0	3.60×10^{-8}	3.09×10^{-06}	7.2×10^{-8}	2.96×10^{-06}
2	1.14×10^{-8}	1.38×10^{-06}	2.0×10^{-8}	2.15×10^{-06}
4	6.72×10^{-8}	1.22×10^{-06}	6.6×10^{-8}	1.58×10^{-06}
6	7.8×10^{-9}	3.53×10^{-06}	5.9×10^{-8}	2.33×10^{-06}
8	4.51×10^{-8}	9.92×10^{-06}	2.7×10^{-8}	2.81×10^{-06}

Case 2:
Table B: For $u(x,t)$.
For $v(x,t)$.

x_j	$\lvert u_{exact} - u_{HAM}\rvert$	$\lvert u_{exact} - u_{ADM}\rvert$	$\lvert v_{exact} - v_{HAM}\rvert$	$\lvert v_{exact} - v_{ADM}\rvert$
-8	1.34×10^{-5}	5.4757×10^{-03}	5.45×10^{-5}	9.3182×10^{-03}
-6	1.49×10^{-5}	5.4945×10^{-03}	8.16×10^{-5}	9.3170×10^{-03}
-4	3.81×10^{-5}	5.4762×10^{-03}	3.73×10^{-5}	9.3214×10^{-03}
-2	8.44×10^{-5}	5.4824×10^{-03}	4.74×10^{-5}	9.3170×10^{-03}
0	6.60×10^{-5}	5.4702×10^{-03}	4.91×10^{-5}	9.3208×10^{-03}
2	1.78×10^{-5}	5.4778×10^{-03}	2.28×10^{-5}	9.3173×10^{-03}
4	2.51×10^{-5}	5.4789×10^{-03}	1.43×10^{-5}	9.3161×10^{-03}
6	3.64×10^{-5}	5.4792×10^{-03}	5.60×10^{-5}	9.3195×10^{-03}
8	6.41×10^{-6}	5.4786×10^{-03}	1.08×10^{-5}	9.3221×10^{-03}

Conclusion

The homotopy analysis method has been well applied on the non-linear reaction diffusion system of equations of Lotka-Volterratypefor finding the approximate solutions. The relationship made between the exact solutionand the HAMshows that HAM is nearly close to the exact solution, and it is very effectiveand accurate as presented by Table 1, 2, 3 and 4. The obtained numerical values as presented in Table A, and Table B show much accuracy of this method as compared to the ADM. Further if we take $h=-1$, we can obtain the results of ADM as a special case of the HAM.

The HAM has the non-zero auxiliary parameter h, by means of which we can control and adjust the convergence area of the series solutions. Unlike the other numerical methods,itgivesa good degree of accuracy for solving high nonlinear problems. Obviously, it is concluded that the HAM is a very reliable, efficient and powerful tool with the help of which we can solvehighly non-linear problems in science and engineering without any limitations andassumptions.

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