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Parameters Estimation of Nakagami Probability Distribution Using Methods of L.Moments

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Abstract

In communications theory, Nakagami distribution (NKD) is used to model scattered signals that reach a receiver from different paths. In order to use NKD to model a given set of data, we will have to estimate its parameters from the given data. Method of L-Moments (MLM) is being compared with Method of Moments (MOM) for estimating the parameters of NKD. In this study, we have derived its first two L-moments in closed form and estimated its parameters using simulated data. This study shows that estimates based on MLM are better than MOM, not only in small samples but also in large samples. For evaluation purpose, we calculated Root Mean Square Error (RMSE) and Bias using Monte Carlo simulations.

Keywords: mL-Moments, Method of Moments, Nakagami distribution, Parameter estimation.

Introduction

THIS distribution was introduced by M. Nakagami in 1960 [1]. In communications theory, Nakagami distribution (NKD) is mostly used for modeling the fading of radio signals. It has two parameters one is shape parameter (*m*-parameter/fading parameter) and other is scale(*w*) parameter. Itis used to model scattered signals that reach a receiver from different paths. Depending on the thickness of the scatter, the signal will display diverse fading properties. NKD can be reduced to Rayleigh distribution, but gives more control over the extent of the fading. The NKD has also been applied successfully in many other fields as well. For example, Shankar et al. [2,3] found that it performs well in the making the unit hydrographs, which is used to estimate runoff in hydrology and Tsui et al. [4] applied the NKD on ultrasound data. Similarly, Carcolé and Sato [5] and Nakahara and Carcolé [6] have shown the utility of the NKD to deal with the formation of high-frequency seismogram envelopes.

In order to use NKD to model a given set of data, we will have to estimate its parameters from the given data. Shape parameter is important in the sense as its knowledge is required by the receiver for optimal reception of signals in Nakagami fading [7].In the literature more attention has been given in estimation of its shape parameter. Many of the estimators for shape parameter are only the approximations to MLE or MOM estimators [8–10]. Some alternative estimators have been considered and compared by Gaeddert and Annamalai [11], Abdi and Kaveh [12], and Beaulieu and Chen [13]. As we know, solution of the scale parameter, n*w* is trivial if we use MOM and MLE and is equal to unbiased estimator of the scale parameter. In our study we estimated both parameters of NKD through MLM and compare it with MOM estimators. The MLM has been extensively used by many researchers in variety of fields such as engineering, meteorology, quality control and Hydrology. L.Moments introduced by Hosking [14] show many advantages over conventional moments, for example, fitted sample MOM cannot explain complete skewness of the distribution, but fitted sample through MLM explain complete skewness of the distribution [14]. L-moments of a probability distribution exist only if its mean is finite. Asymptotic approximations to sampling distributions are better for L-moments than for ordinary moments. Although moment ratios can be arbitrarily large but sample moment ratios have algebraic bounds [15]. They are robust to outliers present in the data and give a better identification of the parent distribution for a given data sample. Due to these so many advantages of MLM over MOM in this study we drive the expressions of L-moments for NKD and estimated its both parameters through MLM. We make comparison of MLM and MOM in terms of RMSE and Bias for its estimated parameters. Our comparison is based on Monte Carlo simulations.

Method of L.Moments (MLM)

L-moments are summary statistics for probability distributions and data samples. They are analogous to conventional moments. They also provide measures of location, dispersion, skewness, kurtosis, and other aspects of the shape of probability distributions or data samples, but in their computation we use linear combinations of the ordered data values.

Let $X_1, X_2, X_3, \ldots\ldots, X_n$ be a random sample and their corresponding order statistics are $X_{1:n} \leq X_{2:n} \leq X_{3:n} \leq \ldots\ldots \leq X_{n:n}$. The rth Population L-moment defined by Hosking [14] is:

$$\lambda_r = \frac{1}{r}\sum_{k=0}^{r-1} \binom{r-1}{k}(-1)^k \, E\,(X_{r-k:\,n})$$

$$\lambda_1 = E(X_{1:1}) \qquad (1)$$

$$\lambda_2 = \frac{1}{2}\left(E(X_{2:2}) - E(X_{1:2})\right) \qquad (2)$$

$$\lambda_3 = \frac{1}{3}\left(E(X_{3:3}) - 2\,E(X_{2:3}) + E(X_{1:3})\right) \qquad (3)$$

$$\lambda_4 = \frac{1}{4}\left(E(X_{4:4}) - 3\,E(X_{3:4}) + 3E(X_{2:4}) - E(X_{1:4})\right) \qquad (4)$$

$E\,(X_{i:\,r})$ can be written as

$$E\,(X_{i:\,r}) = \frac{r!}{(i-1)!\,(r-i)!}\int_0^1 x(F)^{i-1}(1-F)^{r-i}dF$$

Or

$$= \frac{r!}{(i-1)!\,(r-i)!}\int_{-\infty}^{\infty} x f(x)[F(x)]^{i-1}[1-F(x)]^{r-i}dx$$

L-moments based CV is L-CV = ${}^{\lambda_2}/_{\lambda_1}$ its range is:

$0 < L-CV < 1$ L-skewness denoted by

$\tau_3 = {}^{\lambda_3}/_{\lambda_2}$ *and* its range $0 < |\tau_3| < 1$. L-kurtosis

denoted by $\tau_4 = {}^{\lambda_4}/_{\lambda_2}$ and its range

$(5\tau_3^2 - 1)/_4 < \tau_4 < 1$

Sample L-moments as developed by [16] are as

$$l_r = \frac{1}{r\binom{n}{r}}\sum_{i=0}^{n}\sum_{k=0}^{r-1}(-1)^k \binom{r-1}{k}\binom{i-1}{r-1-k}\binom{n-i}{k} X_{i:n} \qquad (5)$$

NAKAGAMI DISTRIBUTION

The probability density function (PDF) of NKD is given by

$$f(x) = \frac{2m^m}{Gama(m)\,w^m} x^{2m-1} e^{-\left(\frac{mx^2}{w}\right)}, \qquad x \geq 0$$

Where "*m*" is shape parameter also known as fading parameter and "*w*" is scale parameter. Cumulative density function (CDF) is given by

$$F(x) = \frac{\Gamma(m) - \Gamma\left(m, \frac{m\,x^2}{w}\right)}{\Gamma(m)}$$

The k^th population raw moment

$$\mu_k = E(x^k) = \frac{\Gamma\left(m + \frac{k}{2}\right)}{\Gamma(m)}\left(\frac{w}{m}\right)^{k/2}$$

This distribution becomes Rayleigh distribution when $m = 1$ and one-sided Gaussian distribution f $m = {}^1/_2$.

L-MOMENTS OF THE NAKAGAMI DISTRIBUTION.

We drive L-moments of NKD using Eq. (1-4). Population L-moments of NKD are.

$$\lambda_1 = \frac{\Gamma\left(m+\frac{1}{2}\right)\left(\frac{w}{m}\right)^{\frac{1}{2}}}{\Gamma(m)} \qquad (6)$$

$$\lambda_2 = \frac{4^{1-m}\sqrt{w}\left((1+2m)\sqrt{\pi}\,\Gamma(2m) - 4^m\Gamma\left(\frac{1}{2}+2m\right)G\right)}{\sqrt{m}\,(1+2m)\Gamma(m)^2} - \lambda_1 \qquad (7)$$

Where "*G*" is defined as,

$$G = Hypergeometric2F1\left[\frac{1}{2}+m, \frac{1}{2}+\,2m, \frac{3}{2}+m, -1\right]$$

For estimation of its parameters, we just needed first two sample l-moments. Although, we can drive 3^rd & 4^th L-moment numerically.

For estimates of parameters we solve the following equations. For shape parameter "*m*",

$$\frac{l_2}{l_1} = \frac{\Gamma(\hat{m})}{\Gamma\left(\hat{m}+\frac{1}{2}\right)\sqrt{\frac{1}{\hat{m}}}}\left(\frac{4^{1-\hat{m}}\left((1+2\hat{m})\sqrt{\pi}\Gamma[2\hat{m}] - 4^{\hat{m}}\Gamma\left[\frac{1}{2}+2\hat{m}\right]\hat{G}\right)}{\sqrt{\hat{m}}(1+2\hat{m})\Gamma[\hat{m}]^2} - \frac{\Gamma\left(\hat{m}+\frac{1}{2}\right)\left(\frac{1}{\hat{m}}\right)^{\frac{1}{2}}}{\Gamma(\hat{m})}\right) \qquad (8)$$

For the specific value of estimate of *m* –parameter, the right hand side of above equation will be equal to left hand side. We can get the sample estimate of scale parameter "*w*" as follows

$$\hat{w} = \left(\frac{l_1\,\Gamma(\hat{m})}{\Gamma\left(\hat{m}+\frac{1}{2}\right)\sqrt{\frac{1}{\hat{m}}}}\right)^2 \qquad (9)$$

MONTE CARLO SIMULATION & DISCUSSION

For the comparison of two methods of moment's estimation we conducted the Monte Carlo experiment. One of the advantages of Monte Carlo simulation as compared to the other numerical methods that can solve the same problem, is that it is conceptually very simple. It does not require any specific knowledge of the form for the solution at hand. Simulation are used to mimic the behavior of real world system. It is a theoretical approach to develop theoretical outputs based on varying input data. We compared their RMSE and bias. We used Mathematica 9 and R-Language software for this purpose. We assumed different parametric value for NKD, and performed our experiment with different sample size (25, 50,100,250). We repeated each of our experiment 10,000 times. Finally, we get the RMSEs& bias for estimated parameters as follows:

RMSE = $\sqrt{\frac{\sum_{i=1}^{N}(x_i-y_i)^2}{N}}$, and Bias= $\frac{\sum_{i=1}^{N}(x_i-y_i)}{N}$:

Where, y_i are fitted values & x_i are observed values. Results are presented in Table 5.1, 5.2& 5.3. We take the parametric values *m=2,3,5& w = 1.5,2.5,4.5*. In most of the cases (for small sample sizes as well as for large samples sizes) these two quantities i.e. RMSE and bias, are less if we use MLM rather than MOM. In some cases, bias is near to zero which implies that we get unbiased estimate of the parameters using MLM in these cases. Our results show that as we increase the value of scale parameter "*w*" values of RMSE & Bias increased in case of MOM estimation, but in case of MLM estimation values of RMSE did not affect too much. On the basis of minimum RMSE and biases, it is found that that L-moments provide better estimates for NKD parameters as compared to conventional moments.

Table-1 Values of RMSE and Bias when "*w*" =1.5

		n = 25		n = 50		n = 100		n = 250	
m		MOM	MLM	MOM	MLM	MOM	MLM	MOM	MLM
2	RMSE	0.8075	0.4157	0.8006	0.4315	0.8160	0.4155	0.8293	0.4131
	Bias	0.2128	0.0079	0.2231	-0.0027	0.2175	0.0171	0.2575	-0.0184
3	RMSE	0.8014	0.3554	0.8285	0.3383	0.8083	0.3419	0.7618	0.3291
	Bias	0.2150	-0.0016	0.1678	0.0004	0.2129	-0.0029	0.2851	0.0015
5	RMSE	0.7620	0.2705	0.7634	0.2832	0.7648	0.2638	0.8185	0.2602
	Bias	0.2132	-0.0100	0.2388	0.0394	0.2301	0.0054	0.1755	-0.0059

Table-2 Values of RMSE and Bias when "*w*" =2.5

		n = 25		n = 50		n = 100		n = 250	
m		MOM	MLM	MOM	MLM	MOM	MLM	MOM	MLM
2	RMSE	1.3554	0.5350	1.4429	0.5453	1.4116	0.5318	1.4142	0.5179
	Bias	0.6183	0.0221	0.4916	-0.0291	0.4786	0.0021	0.5166	0.0031
3	RMSE	1.3707	0.4558	1.3631	0.4430	1.4430	0.4502	1.3578	0.4473
	Bias	0.5878	0.0284	0.6169	0.0014	0.5080	-0.0179	0.5312	0.0067
5	RMSE	1.3746	0.3369	1.4663	0.3415	1.3742	0.3552	1.3825	0.3414
	Bias	0.5717	0.0019	0.4804	0.0004	0.5561	-0.0030	0.5145	-0.0085

Table-3 Values of RMSE and Bias when "*w*" =4.5

		n = 25		n = 50		n = 100		n = 250	
m		MOM	MLM	MOM	MLM	MOM	MLM	MOM	MLM
2	RMSE	2.1731	0.7572	2.0761	0.7449	2.1913	0.7182	2.1434	0.7367
	Bias	1.0366	-0.0021	1.1475	-0.0288	1.0258	0.0021	1.0813	0.0020
3	RMSE	2.1038	0.5998	2.2497	0.6026	2.0627	0.5860	2.2605	0.6188
	Bias	1.0523	0.0082	0.9382	-0.0313	1.0734	-0.0003	1.0750	0.0055
5	RMSE	2.2005	0.4939	2.1206	0.4766	2.0759	0.4656	2.2914	0.4859
	Bias	1.1083	-0.0013	1.1414	0.0072	1.2328	0.0016	1.0236	0.0011

Conclusion

We have derived L-moments of NKD and compared the estimates its parameter using MLM and MOM. Our comparison is based on RMSE & Bias, using simulated data. The comparison shows that performance of L-moments is better than conventional moments. We found that for small samples as well as large samples, L-moments have smaller RMSE and bias. So we can conclude that L-moments estimation is better for parameter estimation of NKD as compared to MOM.

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